# Topological Entropy Conjecture

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**Abstract:** Defines $f-\check{C}ech$ homology group $H_i(X,f;G)$, $f$ is a self-mapping on a compact Hausdorff space $X$. Denotes $f_*$ is the linear transformation induced by $f$ on the homology group $H_*(X;G)=\oplus\sum_{i=0}^{n}H_i(X;G)$. Let $E_{f_*H_*}$ is the set of $f_*$'s eigenvalue on $H_*$, $\rho=\sup\{\|y\| | y\in E_{f_*H_*}\}$, then we have $ent(f_L)\geq\log\rho$,where $ent(f_L)$ is $f$'s topological fiber entropy, and in this sense the Topological Entropy Conjecture is true.

**Keywords:** $\check{C}ech$ homology,equation,germ,fiber entropy,eigenvalue,category.

## 1.Introduction

1974,In[1] Shub stated a conjecture,named Topological Entropy Conjecture, that is:

Let $f\in C^0(M^n)$, $M^n$ is a $n-$ dimension compact manifold and $C^0(M^n)$ is the set of all continuous self-mapping on $M^n$, so $f$ induces a homomorphism $f_*$ on the homology group $f_*:H_*(M^n;Z)\longrightarrow H_*(M^n;Z)$, where $H_*(M^n;Z)=\oplus\sum_{i=0}^{n}H_i(M^n;Z)$, $H_i(M^n;Z)$ is the $i$th homology group with integer coefficients.

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Now $f_*$ is a linear transformation and is a $(n+1)\times(n+1)$ matrix concerning integer entry, let $E_{f_*}$ is the set of $f_*$'s eigenvalue, $\rho = \sup\{\|y\| | y \in E_{f_*}\}$, then the topological entropy conjecture is the inequality $ent(f) \geq \log\rho$ , where $ent(f)$ is $f$'s topological entropy.

The inequality is so simple connected in the first place with the work of Smale,Shub,and Sullivan,that one attempts to prove it have been very fruitful. But unlike the equality of Gauss-Bonet theorem, unlucky,in[4] there is a example explaining the inequality invalid:$0 = ent(f) < \log\|E_{f_* H_*(X;G)}\|$.

After study the counterexamples ,the normal entropy definition and the $\check{C}ech$ cohomology definition, through a little revise in homology define and entropy define,following new definition I proof the conjecture is valid again on compact Hausdorff space, and in this paper omit the proof of obvious lemma or conclusion.

To establish the inequality is my interest, also in this paper there is a lot of words about the homology, and in the end develop the normal homology to $\check{C}ech$ homology, extend the dual theorem between normal homology and $\check{C}ech$ cohomology to the dual theorem between $\check{C}ech$ homology and $\check{C}ech$ cohomology.

For ignore reading the origin paper, there is a story between $Klein$ and *Poincaré* about the naming of $Fuchs$ function in two dimension, course,in the last *Poincaré* was remedial this by named $Klein$ group after his own achievement in three dimension.

Some times I maybe forgot or ignore the references for never reading the origin paper, if you find something,please forgive me and chase me,I will remedy that in the first time.

Generally,in this paper,shift is one-side shift,$id$ or $I$ is the identical mapping,$X$ denotes a compact Hausdorff space, $C^0(X)$ denotes the set of all continuous self-mapping on $X$, $G$ is a free abelian group with a finite spanning set, $Z$ is the

integer set,$Z^n = \bigoplus_{i=1}^{n} Z$ and $Q$ is the rational set. For brevity,$n$ and $\| \bullet \|$ denotes of many kinds meaning, one can regard this following the context.

Let $\alpha, \beta$ is the open cover of $X$, if for any $B \in \beta$,exist $A \in \alpha$ and $B \subseteq A$,define $\alpha < \beta$,named $\beta$ is larger than $\alpha$. Put $\alpha^c = \{A|A^c \in \alpha\}$,where,$A^c \bigcup A = X, A^c \bigcap A = \emptyset$,and $\overline{A}$ is the closures of $A$, $\|A\|$ is the numbers of elements of $A$.

$$\text{Denotes: } \begin{cases} a_0 \cdots \hat{a}_i \cdots a_p = a_0 \cdots a_{i-1}, a_{i+1}, \cdots a_p, \\ a_0 \cdots \underline{b}^{(i)} \cdots a_p = a_0 \cdots a_{i-1}, b, a_i, \cdots a_p, \\ a_0 \cdots \underline{b}^{(i)}_{(k)} \cdots a_p = \sum\limits_{m \in (k)} a_0 \cdots a_{i-1}, b_m, a_i, \cdots a_p, \\ a_0 \cdots \underline{b}^{(i)}_{\emptyset} \cdots a_p = \sum\limits_{m \in \emptyset} a_0 \cdots a_{i-1}, b_m, a_i, \cdots a_p = a_0 \cdots a_{i-1}, a_i \cdots a_p, \\ (k) = \{k_1, k_2, k_3, \cdots, k_n, n = \|\{a_0 \cdots a_{i-1}, b_m, a_i, \cdots a_p\}\| \geq 1, m \in Z\}, \\ (a_0 \cdots \hat{b}_{(k)} \cdots a_p)^d = b_{k_1} \cdots b_{k_i} \cdots b_{k_n}, k_i \in (k). \end{cases}$$

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## 2.Algebra Equation of Boundary Operator

**Definition 1.** [2]$P_{541}$ *Let $\Psi$ is a cover of $X$, Let $U_0, U_1, U_2, \cdots, U_p \in \Psi$, if $U_0 \bigcap U_1 \bigcap \cdots \bigcap U_p \neq \emptyset$, then define a p-simplex $\sigma_p$ and pth chain group $C_p$, so get the pth homology group $H_p(\Psi; G)$ and cohomology group $H^p(\Psi; G)$.*

*where :* $\cdots \to C_{p+1}(\Psi; G) \xrightarrow{\partial_{p+1}} C_p(\Psi; G) \xrightarrow{\partial_p} C_{p-1}(\Psi; G) \to \cdots$

$$\partial_p(U_0 \bigcap \cdots \bigcap U_p) = \sum_{i}^{p} (-)^i (U_0 \bigcap \cdots \bigcap \hat{U}_i \cdots \bigcap U_p),$$

*it is easy to get* $\partial_{p-1} \circ \partial_p = 0$.

$B_p(\Psi;G) = im\partial_{p+1},$

$Z_p(\Psi;G) = ker\partial_p,$

$H_p(\Psi;G) = Z_p/B_p,$

*Let* $C^p(\Psi;G) = Hom(C_p(\Psi;G), G),$

*so* $\partial_p$ *induce a homomorphism* $C^{p-1}(\Psi;G) \xrightarrow{\delta^p} C^p(\Psi;G)$*, and get:*

$$\cdots \leftarrow C^{p+1}(\Psi;G) \xleftarrow{\delta^{p+1}} C^p(\Psi;G) \xleftarrow{\delta^p} C^{p-1}(\Psi;G) \leftarrow \cdots,$$

$$\delta^{p+1} \circ \delta^p = 0.$$

$B^p(\Psi;G) = im\delta^p,$

$Z^p(\Psi;G) = ker\delta^{p+1},$

$H^p(\Psi;G) = Z^p(\Psi;G)/B^p(\Psi;G),$

**Lemma 1.** $C^p(\Psi;G) \cong C_p(\Psi;G)$*, if* $c_p = U_0 \bigcap \cdots \bigcap U_p \in C_p(\Psi;G)$*,put* $U^i = U_i^c$ *then* $c^p = U^0 \bigcup \cdots \bigcup U^p \neq X$ *is a representation of p-simplex of* $C^p(\Psi;G)$*.*

*Proof.* Because of finite spanning and free, $G$ can be regard as a ring [3], so $C_p(\Psi;G)$ can be treated as a finite dimension $G$-module vector space[7], and $C^p(\Psi;G)$ can be treated as the dual vector space of $C_p(\Psi;G)$. With the property of finite dimension $G$-module vector space, can get $C^p(\Psi;G) \cong C_p(\Psi;G)$.

$$c_p = U_0 \bigcap \cdots \bigcap U_p \neq \emptyset \longleftrightarrow c^p = U^0 \bigcup \cdots \bigcup U^p \neq X,$$

so $c_p \in C^p(\Psi;G) \Longleftrightarrow c^p \in C^p(\Psi;G)$, i.e.,$U^0 \bigcup \cdots \bigcup U^p \neq X$ is a representation of $p$-simplex of $C^p(\Psi;G)$. □

**Definition 2.** *Let* $X$ *is a compact space, define* $n_\Psi = \max\{n|\partial(U_0 \cdots \bigcap U_i \cdots U_n) \neq \partial(U_0 \cdots \bigcap U_i \cdots U_n \bigcap U_{n+1}), U_1, \cdots U_i, \cdots, U_n, U_{n+1} \in \Psi\}$*, named* $n_\Psi$ *is the* $\partial$ *operator dimension of* $X$ *with* $\Psi$*,where* $\Psi$ *is a cover of* $X$*. Let* $J$ *is a orientation set induced by the all open cover of* $X$*, easily to find that :if* $\alpha, \beta \in J, \alpha < \beta$*, then* $n_\alpha \leq n_\beta$*. so* $n$ *is a function defined on* $J$*, i.e.,*$n_\Psi = n(\Psi)$*. If exist* $\varinjlim_{\Psi \in J} n_\Psi = n_J = n(J)$*,then define* $n_J$ *is the* $\partial$ *operator dimension of* $X$*.*

Because $n$ is a function defined on $J$, for convenience $n$ denotes the value $n(\Psi)$, so its can be $n = n_\Psi$,can be $n = n_J$,some particular places using $n$ denotes normal manifold dimension too,one can regard this following the context. In order to deal with the problem easily, always let $\Psi \in J$ is good enough and enough refinement,i.e., satisfy all the necessary requirements of the problem.

**Lemma 2.** *For any $\sigma^p$, exist a $\sigma^n$, that is, $\sigma^p$ is the pth surface of $\sigma^n$, where $p \leq n, \sigma^p \in C^p(\Psi; G), \sigma^n \in C^n(\Psi; G)$, so exist $(U^0 \bigcup \cdots \bigcup \hat{U_{(k)}} \cdots \bigcup U^p)^d = U^{k_0} \bigcup \cdots \bigcup U^{k_m} \cdots \bigcup U^{k_{n-p+1}}$.*

*Proof.* if $p = n$,the conclusion is trivial,

put $p < n$ and the exist of $\sigma^n$ is following the define of $n$,

let $\sigma^n = V^0 \bigcup \cdots \bigcup V^n, \sigma^p = U^0 \bigcup \cdots U^i \cdots \bigcup U^p, p < n, \sigma^p \in C^p(\Psi; G), \sigma^n \in C^n(\Psi; G)$.

loss no generally let $V^k \neq U^j, 0 \leq k \leq n, 0 \leq j \leq p$,

considering $U^{p+1}, U^0 \bigcup \cdots U^i \cdots \bigcup U^{p+1} = X, 1 \leq i \leq p$.

by the define of $n$,also get $V^0 \bigcup \cdots \bigcup V^n \bigcup U^{p+1} = X$,

then $V^0 \bigcup \cdots \bigcup V^n \bigcup U^{p+1} = U^0 \bigcup \cdots U^i \cdots \bigcup U^{p+1}$,

that is $|\sigma^n| = |\sigma^p|$,i.e.,$\sigma^p$ can be refinement by some $\sigma^n$, where $|\sigma^n|$ is the support set of $\sigma^n$.

but $C^p(\Psi; G) \cong C_p(\Psi; G), C^n(\Psi; G) \cong C_n(\Psi; G)$, so $U^0 \bigcup \cdots \bigcup U^p \neq X$ also is a representation of $p$-simplex of $C_p(\Psi; G)$,

by the property of $Hom(-, G)$ functor,exist a injective homomorphism $f$ from $C^p(\Psi; G)$ to $C^n(\Psi; G)$,

i.e.,$\sigma^p$ can be regarded as a surface of $\sigma^n$,

so exist $(U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p)^d=U^{k_1}\bigcup\cdots\bigcup U^{k_m}\cdots\bigcup U^{k_{n-p+1}}$. $\square$

**Lemma 3.** $H^p(\Psi;G)\cong H_{n-p}(\Psi;G)$.

*Proof.* By *lemma* 1 get two chains:

$$(1)\begin{cases}\cdots\to C_{p+1}(\Psi;G)\xrightarrow{\partial_{p+1}}C_p(\Psi;G)\xrightarrow{\partial_p}C_{p-1}(\Psi;G)\to\cdots\\ \cdots\leftarrow C^{p+1}(\Psi;G)\xleftarrow{\delta^{p+1}}C^p(\Psi;G)\xleftarrow{\delta^p}C^{p-1}(\Psi;G)\leftarrow\cdots\end{cases}$$

for a fixed $p$-simplex of $C_p(\Psi;G)$,considered the algebra equation:

$$(*)\begin{cases}<\partial c_p,c^{p-1}>=<c_p,\delta c^{p-1}>,\\ \partial_p(U_0\bigcap\cdots\bigcap U_p)=\sum\limits_{i=0}^{p}(-1)^i(U_0\bigcap\cdots\bigcap\hat{U}_i\cdots\bigcap U_p),\\ \partial\emptyset=\delta\emptyset=0,\\ <a,\emptyset>=<\emptyset,b>=0\end{cases}$$

$$(**)\begin{cases}<\sum\limits_{i=0}^{p}(-1)^i(U_0\bigcap\cdots\bigcap\hat{U}_i\cdots\bigcap U_p),U_0\bigcap\cdots\bigcap\hat{U}_i\cdots\bigcap U_p>=<c_p,\delta c^{p-1}>,\\ \sum\limits_{i=0}^{p}(-)^i(U^0\bigcup\cdots\bigcup\underline{U_{(k)}^{(i)}}\cdots\bigcup U^p)=\delta^p(U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p),\end{cases}$$

if $(k)=\emptyset$, then define $(i)=\emptyset$,$(-1)^{\emptyset}=0$ ,so $\delta^p(U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p)=0$.

$$(2)\begin{cases}\partial_p(U_0\bigcap\cdots\bigcap U_p)=\sum\limits_{i=0}^{p}(-1)^i(U_0\bigcap\cdots\bigcap\hat{U}_i\cdots\bigcap U_p),\\ \delta^p(U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p)=\sum\limits_{i=0}^{p}(-1)^i(U^0\bigcup\cdots\bigcup\underline{U_{(k)}^{(i)}}\cdots\bigcup U^p).\end{cases}$$

that is :

$$(3)\begin{cases}\partial_p(U_0\bigcap\cdots\bigcap U_p)-\sum\limits_{i=0}^{p}(-1)^i(U_0\bigcap\cdots\bigcap\hat{U}_i\cdots\bigcap U_p)=0,\\ \delta^p(U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p)-\sum\limits_{i=0}^{p}(-1)^i(U^0\bigcup\cdots\bigcup\underline{U_{(k)}^{(i)}}\cdots\bigcup U^p)=0,\\ \delta^{n-p+1}(U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p)^d=\delta^{n-p+1}(U^{k_1}\bigcup\cdots\bigcup U^{k_m}\cdots\bigcup U^{k_{n-p+1}}),\\ \delta^{n-p+1}(U^{k_1}\bigcup\cdots\bigcup U^{k_m}\cdots\bigcup U^{k_{n-p+1}})-\sum\limits_{i=0}^{p}(-1)^i(U^{k_1}\bigcup\cdots\bigcup\underline{U_{(k)}^{(i)}}\cdots\bigcup U^{k_{n-p+1}})=0.\end{cases}$$

let $c_p=\sum z_m(U_0\bigcap\cdots\bigcap U_p)_m$ , so $c^{n-p}=\sum z_m((U^0\bigcup\cdots\bigcup\hat{U_{(k)}}\cdots\bigcup U^p)^d)_m$,

$z_m \in Z.$

$$(*3)\begin{cases} U_0\bigcap\cdots\bigcap U_p \longleftrightarrow U^0\bigcup\cdots\bigcup \hat{U_{(k)}}\cdots\bigcup U^p \longleftrightarrow (U^0\bigcup\cdots\bigcup \hat{U_{(k)}}\cdots\bigcup U^p)^d, \\ c_p \in ker\partial_p \Longleftrightarrow c^{n-p} \in ker\delta^{n-p+1}, \\ c_p \in im\partial_{p+1} \Longleftrightarrow c^{n-p} \in im\delta^{n-p}. \end{cases}$$

$$\text{Denotes: } (4)\begin{cases} \partial_{\frac{ker}{im}}(C_p) = H_p(\Psi;G) = Z_p/B_p = ker\partial_p/im\partial_{p+1}, \\ \partial^*_{\frac{ker}{im}}(C^p) = H^p(\Psi;G) = Z^p/B^p = ker\delta^{p+1}/im\delta^p. \end{cases}$$

So $\partial_p$ and $\delta^{n-p+1}$ is dual solution in algebra equation (3), corresponding $\partial_{\frac{ker}{im}}$ and $\partial^*_{\frac{ker}{im}}$ is dual value in $(*3)$, the all process of dual mapping is linear reversible, i.e.,the same style as homeomorphism,therefore, the $p$th value of $\partial_{\frac{ker}{im}}$ in the $C_p$ chain group is isomorphic to the $(n-p)$th value of $\partial^*_{\frac{ker}{im}}$ in the $C^{n-p}$ chain group, that is $\partial_{\frac{ker}{im}}(C_p) \cong \partial^*_{\frac{ker}{im}}(C^{n-p})$, for this reason,$H^p(\Psi;G) \cong H_{n-p}(\Psi;G)$. □

Like linear equation,let $S_i : A_ix + B_iy + C_iz = 0$ is a collection of lines, in other way is a collection of planes $S^*_i : A_ix + B_iy + C_iz = 0$, where $0 \leq i \leq n$. Line and plane is a pair of dual, but for a fixed space,the value never changed,of intrinsic relationship between line or between plane, i.e.,if $f, g$ are mappings,denotes $f_i = f(S_i, S_{i-1}), f^*_i = f^*(S^*_i, S^*_{i+1})$, and if $g_i = g(f_i), g^*_i = g(f^*_i)$, then $g_i, g^*_i$ is a pair of dual,that is exist a natural relate between $g_i$ and $g^*_{n-i}$. For example, that natural relation maybe is $g_i = g^*_{n-i}$, or $g_ig^*_{n-i} = 1$, or $g_i + g^*_{n-i} = 0$,or $g_iA_k + g_{n-i}B_k + C_k = 0$,and so on,the dual outcome and the representation of the natural relate between $g_i$ and $g^*_{n-i}$ only dependent the dual mappings $f, g$.

## 3.Germ and Dual of *Čech* homology

**Definition 3.** [2]$P_{542}$ *Let $J$ is a orientation set induced by the all open cover of $X$, Let $U_0, U_1, U_2, \cdots, U_p \in \Psi, \Psi \in J$, if $U_0\bigcap U_1\bigcap\cdots\bigcap U_p \neq \emptyset$, then define a $p$-simplex $\sigma_p$ and $p$th chain group $C_p$, so get the $p$th homology group $H_p(\Psi;G)$ and cohomology group $H^p(\Psi;G)$. If $\Omega, \Psi \in J$ and $\Omega < \Psi$, then get homomorphism*

$f_{\Psi\Omega} : H_p(\Psi; G) \to H_p(\Omega; G)$, *and* $f_{\Omega\Psi} : H^p(\Omega; G) \to H^p(\Psi; G)$.

**Definition 4.** *with* $definition$ 3, *If* $\Omega, \Psi \in J$,*denotes* $\Theta = \Psi \bigvee \Omega$, *then* $f_{\Theta\Omega} : H_p(\Theta; G) \to H_p(\Omega; G)$, *and* $f_{\Theta\Psi} : H_p(\Theta; G) \to H_p(\Psi; G)$, *by this,define a Čech homology germ* $H_p(J; G)$. *Also can define a Čech cohomology germ* $H^p(J; G)$. *If for any* $\Psi \in J$,*then* $H_{(n-p)}(\Psi; G) \cong H^p(\Psi; G)$, *define* $H^p(J; G) \cong H_{(n-p)}(J; G)$.

**Definition 5.** [2] $P_{542}$ *Let* $J$ *is a orientation set induced by the all open cover of* $X$, *Let* $U_0, U_1, U_2, \cdots, U_p \in \Psi, \Psi \in J$, *if* $U_0 \bigcap U_1 \bigcap \cdots \bigcap U_p \neq \emptyset$, *then define a p-simplex* $\sigma_p$ *and pth chain group* $C_p$, *so get the pth homology group* $H_p(\Psi; G)$ *and cohomology group* $H^p(\Psi; G)$. *If* $\Omega, \Psi \in J$ *and* $\Omega < \Psi$, *then get homomorphism* $f_{\Psi\Omega} : H_p(\Psi; G) \to H_p(\Omega; G)$, *and* $f_{\Omega\Psi} : H^p(\Omega; G) \to H^p(\Psi; G)$. *finally define Čech pth cohomology group* $\check{H}^p(X; G) = \varinjlim_{\Omega\in J} H^p(\Omega; G)$.

**Definition 6.** *With* $definition$ 5, *define Čech pth homology group* $H_p(X; G) = \varprojlim_{\Omega\in J} H_p(\Omega; G)$.

**Lemma 4.** $H_p(X; G) \sim H_p(J; G)$, $\check{H}^p(X; G) \sim H^p(J; G)$,*where '*$\sim$*' means different expressions of the same thing.*

*Proof.* By $lemma$ 3 and $definition$ 4, 5, 6. □

**Definition 7.** *If* $H_{(n-p)}(J; G) \cong H^p(J; G)$,*then define* $H_{(n-p)}(X; G) \cong \check{H}^p(X; G)$.

**Theorem 1.** $H_{(n-p)}(X; G) \cong \check{H}^p(X; G)$.

*Proof.* By $lemma$ 4,$definition$ 7. □

## 4.$f$ − *Čech* homology and $L_1$ Category

**Definition 8.** *Let* $U_i, V, W \subseteq X, 0 \leq i \leq k, k \in Z$,$f \in C^0(X)$. *define:*

$$
\begin{aligned}
&L_f(U)=\{\cdots,f^{-n}(U),\cdots,f^{-1}(U),f^0(U),\cdots,f^n(U),\cdots\},\\
&f\circ L_f=L_f\circ f,\\
&L_f(U)\bigcap L_f(V)=L_f(W),\qquad W=U\bigcap V,\\
&L_f(\emptyset)=\emptyset,\\
&L_f(U_0)\bigcap L_f(U_1)\cdots\bigcap L_f(U_k)=L_f(U_0)\bigcap(L_f(U_1)\cdots\bigcap L_f(U_k)),\\
&L_{g+h}(U)=\{\cdots,g^{-n}(U)\bigcup h^{-n}(U),\cdots,g^0(U)\bigcup h^0(U),\cdots,g^n(U)\bigcup h^n(U),\cdots\},\\
&L_{g\bigoplus h}(U)=L_{g+h}(U),when\, g^{-1}(U)\bigcap h^{-1}(U)=\emptyset,\\
&f|_U=g+h\Longleftrightarrow L_f(f(U))=L_{g+h}(f(U)),\\
&f=g+h\Longleftrightarrow L_f(U)=L_{g+h}(U).
\end{aligned}
$$

*then $L_f(U)$ is named the $f$-fiber of $U$,let $X^f$ denotes the sets of $L_f(U)$.*

If $X$ is a compact space,then $X^{+\infty}=\prod\limits_{i=-\infty}^{-1}X\times\underline{X}\times\prod\limits_{i=1}^{+\infty}X$ is compact too,by Tychonoff theorem. Clearly,$X^f$ is a compact subset of $X^{+\infty}$.

**Definition 9.** *Let $J$ is a orientation set induced by the all open cover of $X$, $\Psi\in J$,$U_0,U_2,\cdots,U_p\in\Psi$,$f\in C^0(X)$. if $L_f(U_0)\bigcap\cdots\bigcap L_f(U_p)\neq\emptyset$, then define a $(\Psi,f)$ p-simplex $\check{\sigma}_p$, and get the $(\Psi,f)$ pth homology group $H_p(\Psi,f;G)$.*

**Lemma 5.** *A Čech p-chain $c_p$ can induce a $f-$Čech p-chain $\check{c}_p$, and so Čech p-chain group is isomorphic to a subgroup of $f-$Čech p-chain group.*

*Proof.* □

**Definition 10.** *Let $J$ is a orientation set induced by the all open cover of $X$, $\Psi\in J$,$U_0,U_2,\cdots,U_p\in\Psi$,$f\in C^0(X)$, define: $\check{\partial}_p:C_p(\Psi,f;G)\to C_{p-1}(\Psi,f;G)$,*

$$\check{\partial}_p(L_f(U_0)\bigcap\cdots\bigcap L_f(U_p))=\sum_{i=0}^{p}(-1)^i(L_f(U_0)\bigcap\cdots\bigcap L_f(\hat{U_i})\cdots\bigcap L_f(U_p)).$$

*it is easy to get $\check{\partial}\circ\check{\partial}=0$.*

*in fact $\check{\partial}\circ\check{\partial}(L_f(U_0)\bigcap\cdots\bigcap L_f(U_p))$*

$$=\sum_{i}^{p}(-1)^i\check{\partial}(L_f(U_0)\bigcap\cdots\bigcap L_f(\hat{U_i})\cdots\bigcap L_f(U_p))$$

$$= \sum_{i}^{p} \sum_{j<i} (-1)^{i+j} (L_f(U_0) \bigcap \cdots \bigcap L_f(\hat{U_j}) \cdots \bigcap L_f(\hat{U_i}) \cdots \bigcap L_f(U_p))$$

$$+ \sum_{i}^{p} \sum_{j>i} (-1)^{i+j-1} (L_f(U_0) \bigcap \cdots \bigcap L_f(\hat{U_i}) \cdots \bigcap L_f(\hat{U_j}) \cdots \bigcap L_f(U_p))$$

$$= 0.$$

Similarly we get the $f - \check{C}ech$ $p$-chain group $C_p(\Psi, f; G)$,and the homomorphism sequence:

$$\cdots \to C_{p+1}(\Psi, f; G) \xrightarrow{\check{\partial}_{p+1}} \check{C}_p(\Psi, f; G) \xrightarrow{\check{\partial}_p} C_{p-1}(\Psi, f; G) \to \cdots$$

$$\text{and} \quad \begin{aligned} &B_p(\Psi, f; G) = im\check{\partial}_{p+1}, \\ &Z_p(\Psi, f; G) = ker\check{\partial}_p, \\ &H_p(\Psi, f; G) = Z_p(\Psi, f; G)/B_p(\Psi, f; G), \end{aligned}$$

**Definition 11.** *With definition* 9,*If* $\Omega, \Psi \in J$ ,$\Omega < \Psi$, *then get a homomorphism* $f_{\Psi\Omega} : H_p(\Psi, f; G) \to H_p(\Omega, f; G)$*, finally define* $f - \check{C}ech$ *pth homology germ* $H_p(J, f; G)$ *and pth homology group* $H_p(X, f; G) = \varprojlim_{\Omega \in J} H_p(\Omega, f; G)$.

**Lemma 6.** $H_p(X, J; G) \sim H_p(X, f; G)$.

*Proof.* □

$$\text{define:} \quad \begin{aligned} &C_p(X; G) = \bigoplus C_p(X, id; G), \\ &H_*(X; G) = \bigoplus_{i=0}^{n} H_i(X; G), \\ &C_*(X; G) = \bigoplus_{i=0}^{n} C_i(X; G), \\ &B_*(X; G) = \bigoplus_{i=0}^{n} B_i(X; G), \\ &H_*(X, f; G) = \bigoplus_{i=0}^{n} H_i(X, f; G), \\ &C_*(X, f; G) = \bigoplus_{i=0}^{n} C_i(X, f; G), \\ &B_*(X, f; G) = \bigoplus_{i=0}^{n} B_i(X, f; G), \end{aligned}$$

**Lemma 7.** *Let $G$ is a abelian group, $G_0$ is a subgroup of $G$, then a linear transformation $T$ on $G$ restricted on $G_0$ noted by $T_0$,also is a linear transformation. if define $E_T$ is the all eigenvalue of $T$,so is $E_{T_0}$, then $\sup\|E_T\| \geq \sup\|E_{T_0}\|$, where $\|E_T\| = \{\|a\| | a \in E_T\}$. Denotes $f_*H_*$ is the induced linear transformation of $f$,restricting on $H_*$, in this obtaining the inequality:*

$$
\begin{aligned}
&\sup\|E_{f_*H_*(X,f;G)}\| \leq \sup\|E_{f_*Z_*(X,f;G)}\| \leq \sup\|E_{f_*C_*(X,f;G)}\|,\\
(5)\quad &\sup\|E_{f_*H_*(X;G)}\| \leq \sup\|E_{f_*Z_*(X;G)}\| \leq \sup\|E_{f_*C_*(X;G)}\|,\\
&\sup\|E_{f_*C_*(X;G)}\| \leq \sup\|E_{f_*C_*(X,f;G)}\|,
\end{aligned}
$$

*Proof.* □

**Definition 12.** *$X, Y$ is compact Hausdorff spaces,$f \in C^0(X), g \in C^0(Y)$.*

*(a)Define $X^f$ and $Y^g$ is $L_1$-homotopy equivalence, if exist a pair of continuous mapping:*

$$
\begin{aligned}
&F : X^f \to Y^g, \quad D : Y^g \to X^f,\\
&F \circ D = id_{Y^g}, \quad D \circ F = id_{X^f}.
\end{aligned}
$$

*(b)Define $h, r : X^f \to Y^g$ is $L_2$-homotopy, if exist a continuous mapping:*

$$
\begin{aligned}
&F : X^f \times [0,1] \to Y^g,\\
&F(X^f, 0) = h(X^f), \qquad F(X^f, 1) = r(X^f)
\end{aligned}
$$

*so $h$ induces a homomorphism $h_* : H_{*(X,f;G)} \to H_{*(Y,g;G)}$,and $r_*$ by $r$.*

let $L$ be the class of set:$\{(X^f)|X$ is compact Hausdorff spaces,$f \in C^0(X)\}$, for each $X^f, Y^g \in L$,let $mor_s(X^f, Y^g) = L_1(X^f, Y^g)$, by the $L_1$-homotopy and composition of function $\circ$, we get a category$(L, mor_s, \circ)$.

let $\tilde{L}$ be the class of set:$\{H_{*(X,f;G)}|X^f \in L\}$, for each $H_{*(X,f;G)}, H_{*(Y,g;G)} \in \tilde{L}$, let $mor_H(H_{*(X,f;G)}, H_{*(Y,g;G)})$ be the all group homomorphisms from $H_{*(X,f;G)}$ to $H_{*(Y,g;G)}$, by the $L_1$-homotopy induced $*$ mapping and composition of func-

tion $\circ$, we get a category$(\tilde{L}, mor_H, \circ)$. Easy to see the functor from $(L, mor_s, \circ)$ to $(\tilde{L}, mor_H, \circ)$.

**Theorem 2.** *Let $f \in C^0(X), g \in C^0(Y)$,$X, Y$ is compact Hausdorff spaces,*

*(a)if $X^f$ and $Y^g$ is $L_1$-homotopy equivalence,*

*then $C_p(X, f; G) = C_p(X, g; G)$.*

*(b)if $h, r : X^f \to Y^g$ is $L_2$-homotopy,*

*then $h_* = r_*$.*

*Proof.* By diagram chasing. □

### 5.Topological Entropy Conjecture

if $\alpha, \beta$ are collections of open sets of $X$,

denotes: (6)
$$\begin{aligned}
&\alpha \bigvee \beta = \{A \bigcap B | A \in \alpha, B \in \beta\},\\
&f^{-1}(\alpha) = \{f^{-1}(A) | A \in \alpha\},\\
&f^{-1}(\alpha \bigvee \beta) = f^{-1}(\alpha) \vee f^{-1}(\beta),\\
&\bigvee_{i=0}^{n-1} f^{-i}(\alpha) = \alpha \bigvee f^{-1}(\alpha) \bigvee \cdots \bigvee f^{-(n-1)}(\alpha)
\end{aligned}$$

If $\alpha, \beta$ are open covers of $X$ , let $N(\alpha)$ is the infimum of the numbers of $\alpha' s$ subcover, from the compact of $X$ we know $N(\alpha)$ is a positive integer. so define $H(\alpha) = \log N(\alpha) \geq 0$.

(7)$\alpha < \beta \Longrightarrow H(\alpha) \leq H(\beta)$[8]$P_{81}$.

**Definition 13.** [8]$P_{89}$ *For a fix open cover $\alpha$ of $X$,define :*

*$ent(f, \alpha) = \lim\limits_{n \to \propto} \frac{1}{n} H(\bigvee\limits_{i=0}^{n-1} f^{-i}(\alpha))$,*

*and define:$f' s$ topological entropy:*

$ent(f) = \sup\limits_{\alpha}\{ent(f,\alpha)\}$,*where* $\sup\limits_{\alpha}$ *is through the all open cover of* $X$.

**Lemma 8.** *If* $\alpha$ *is a open cover of* $X$,*let* $L_f(\alpha) = \{L_f(U)|U \in \alpha\}$, *then* $L_f(\alpha)$ *can extend to a open fiber cover* $\dot{L}_f(\alpha)$ *of* $X^f$.

*Proof.* □

**Definition 14.** *For a fix open fiber cover* $\dot{L}_f(\alpha)$ *of* $X^f$, *define :*

$\frac{f^{-1}(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)} = \max\limits_{U\in\alpha} \|\{f^{-1}\dot{L}_f(U) \bigcap \dot{L}_f(U)\}\|,$

$\frac{f(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)} = \max\limits_{U\in\alpha} \|\{f\dot{L}_f(U) \bigcap \dot{L}_f(U)\}\|,$

$L_d = \max\{\frac{f^{-1}(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)}, \frac{f(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)}\},$

$ent(f_L, \dot{L}_f(\alpha)) = ent(f,\alpha) + \log L_d,$

*and define:*$f'$*s topological fiber entropy:*

$ent(f_L) = \sup\limits_{\dot{L}_f(\alpha)} \{ent(f_L, \dot{L}_f(\alpha)) + \log L_d\}$, *where* $\sup\limits_{\dot{L}_f(\alpha)}$ *is through the all open cover of* $X^f$.

If $f \in C^0(X)$,then $f_*$ is the linear transformation induced by $f$, then define *Čech* eigenchains is the chains belong to the eigenvalue following $f_*$, and any *Čech* eigenchains can extend to a open cover.

**Lemma 9.** [8]$P_{102}$ *If* $f$ *is the shift operator on a k-symbolic space,then* $ent(f) = \log k$.

**Lemma 10.** *If* $f$ *is the shift operator on a k-symbolic space,then* $ent(f_L) = ent(f) + L_d = 2\log k$.

*Proof.* □

**Exercise1:**

$$\text{Let } \{1,2,\cdots,k\} = X \text{ ,if } \quad 2^X \xrightarrow[]{f:\ \begin{array}{l}\{1\} \to \{1,2,\cdots,k\},\\ \{2\} \to \{1,2,\cdots,k\},\\ \ \vdots\ \ \vdots \qquad \vdots \\ \{k\} \to \{1,2,\cdots,k\}\end{array}} 2^X,$$

then $ent(f) = 0, ent(f_L) = \log k$.

**Exercise2:**

$$\text{Let } \{1,2,\cdots,k\} = X \text{ ,if } \quad 2^X \xrightarrow{f:\ \{1,2,\cdots,k\} \to \{1\}} 2^X,$$

then $ent(f) = 0, ent(f_L) = \log k$.

**Exercise3:**

$$\text{Let } [0,1] = X \text{ ,if } \quad X \xrightarrow{\begin{array}{c} f(x) = kx, \\ 0 < k < 1, \end{array}} X,$$

then $ent(f) = 0, ent(f_L) = -\log k$.

**Lemma 11.** *Let $2 < m \in Z$,then exist $1 \leq p, q \in Z, p \neq q, m = p + q$.*

*Proof.* □

**Lemma 12.** *Let $\alpha$ is a collection subset of $X$, if $L_f(\alpha)$ is a Čech eigenchains belong to $m$, then $L_f(\alpha)$ has a factor conjunct a shift operator on $m$-symbolic space or its $L_d = m$,$0 \leq m \in Z$.*

*Proof.* If $L_f = \sum_{i=0}^{k} a_i \check{\sigma}_i$ is a eigenchains belong to the eigenvalue $m$, $\check{\sigma}_i \in H_*(X, f; G)$,$m, a_i \in Z$.

with the limit of $H_p(X,f;G) = \varprojlim_{\Omega\in J} H_p(\Omega,f;G)$, there is a $\check{C}ech$ homology germ $H_p(J,f;G)$,$\Phi\in J$,and $H_*(\Phi,f;G)$, so $f_*(L_f) = m(L_f)$ on $H_*(\Phi,f;G)$,it can extend a equation on $C_*(\Phi,f;G)$, that is $f_\sharp(\check{\sigma}_i) = m(\check{\sigma}_i), i\in\{0,\cdots,k\}$, $\check{\sigma}_i\in C_*(\Phi,f;G)$,$m,a_i\in Z$.

Just thinking $f_\sharp$ on $C_*(\Phi,f;G)$,

let $U_0,\cdots,U_j$ is subset of $X$, $\check{\sigma}_i = L_f(U_0)\bigcap\cdots\bigcap L_f(U_j)$, then

$L_f(U_\eta) = \{\cdots,f^{-n}(U_\eta),\cdots,f^{-1}(U_\eta),f^0(U_\eta),\cdots,f^n(U_\eta),\cdots\}, \eta\in\{0,\cdots,j\}$.

so $f_\sharp(\check{\sigma}_i) = f_\sharp(L_f(U_0)\bigcap\cdots\bigcap L_f(U_j)) = L_f(f(U_0))\bigcap\cdots\bigcap L_f(f(U_j)) = m(L_f(U_0)\bigcap\cdots\bigcap L_f(U_j))$,

i.e., $m(\bigcap_{\eta=0}^{j}\{\cdots,f^{-n}(U_\eta),\cdots,f^{-1}(U_\eta),\underline{f^0(U_\eta)},\cdots,f^n(U_\eta),\cdots\})$

$= \bigcap_{\eta=0}^{j}\{\cdots,f^{-n}(f(U_\eta)),\cdots,f^{-1}(f(U_\eta)),\underline{f^0(f(U_\eta))},\cdots,f^n(f(U_\eta)),\cdots\}$

$= \bigcap_{\eta=0}^{j}\{\cdots,f^{-(n-1)}(f(U_\eta)),\cdots,f^{-1}(f(U_\eta)),\underline{f(U_\eta)},f^2(U_\eta),\cdots,f^{n+1}(U_\eta),\cdots\}$

that is $m(\bigcap_{\eta=0}^{j} L_f(U_\eta)) = (\bigcap_{\eta=0}^{j} L_f(f(U_\eta)))$,

loss no generally let $j=0$, get $L_f(f(U_0)) = m(L_f(U_0))$,

$(i), m = 0,1$, the conclusion is trivial,

$(ii), m = 2$, exist $U\subseteq f^{-1}(f(U_0))$,$U\nsubseteq U_0$ and $U_0\nsubseteq U$, $U_0,U$ are non-empty open subsets.

else $f^{-1}(f(U_0)) = U_0$,i.e.$L_f(f(U_0)) = (L_f(U_0)) = 2(L_f(U_0))$, while $G$ is a free group,this is a contradict.

by the property of Hausdorff space,$U\nsubseteq U_0$ and $U_0\nsubseteq U$,$U_0,U$ is open sub-

sets, so exist points $x \in U_0, x \notin U, y \in U, y \notin U_0$ and open neighborhoods $O(x)$ of $x$,$O(y)$ of $y$, where $O(x) \in U_0, O(x) \notin U$ and $O(y) \in U, O(y) \notin U_0$, i.e.,$O(x), O(y) \subseteq f^{-1}(f(U_0))$, $O(x) \bigcap O(y) = \emptyset$.

so when $m = 2$ the conclusion is true.

$(iii) m \geq 3$,

by induction,when $m = n - 1$ the conclusion is right,look $m = n$,

using *lemma* 11,hence $m = p+q, p \neq q, L_f(f(U_0)) = p(L_f(U_0))+q(L_f(U_0))$,

let $f|_{U_0} = h + g, L_h(f(U_0)) = p(L_f(U_0)), L_g(f(U_0)) = q(L_f(U_0))$,

get $L_f(f(U_0)) = L_h(f(U_0)) \bigoplus L_g(f(U_0))$,

else $h^{-1}(f(U_0)) \bigcap g^{-1}(f(U_0)) = W \neq \emptyset$, then $p(L_f(W)) = q(L_f(W))$,but $p \neq q$ contradict with the property of free group.

so $p, q \leq n - 1$,for the induction,

exist $\begin{cases} h^{-1}(f(U_0)) \supseteq U_{0i}, U_{0j}, & U_{0i} \bigcap U_{0j} = \emptyset, 1 \leq i, j \leq p, \\ g^{-1}(f(U_0)) \supseteq U_{1k}, U_{1l} & U_{1k} \bigcap U_{1l} = \emptyset, 1 \leq k, l \leq q, \end{cases}$

where $U_{0i}, U_{0j}, U_{1k}, U_{1l}$ is non-empty open subset.

with $L_f(f(U_0)) = L_h(f(U_0)) \bigoplus L_g(f(U_0))$

get $f^{-1}(f(U_0)) \supseteq U_i, U_j$ and $U_i \bigcap U_j = \emptyset, 1 \leq i, j \leq m, U_i$ is non-empty open subset,

has a mapping $f^{-1}(f(U_0)) \to \{U_k$ is non-empty open subset$|U_i \bigcap U_j = \emptyset, 1 \leq i, j, k \leq m\}$,

and can get a $m$-symbolic space $S_m$ or its $L_d = m$,

so $L_f(U_0)$ has a factor conjunct a shift operator on $S_m$ or its $L_d = m$,

so when $m = n$ the conclusion is right,in the end the conclusion is right for any eigenvalue $m, 0 \leq m \in Z$.

$\square$

**Theorem 3.** *After topological fiber entropy ,the Topological Entropy Conjecture is true on compact Hausdorff space* $X$*,i.e.,* $ent(f_L) \geq \sup \|E_{f_* H_*(X;G)}\|$.

*Proof.* following *lemma* 12 we get:

$$ent(f_L) \geq ent(f_L, \dot{L}_{(}\alpha)) \geq \log \|E_{f_* C_*(X,f;G)}\| \geq \log \|E_{f_* C_*(X;G)}\| \geq \log \|E_{f_* H_*(X;G)}\|.$$

$\square$

**Lemma 13.** $ent(f_L) \geq ent(f)$*,inequality can be strict inequality.*

*Proof.* $\square$

**Lemma 14.** $ent(I_L) = ent(I) = 0$*,*$I$ *is identical mapping.*

*Proof.* $\square$

**Corollary 1.** *In the sense of Čech cohomology,Topological Fiber Entropy is true on compact Hausdorff space* $X$*, i.e.,* $ent(f_L) \geq \log \|E_{f^* \check{H}^*}\|$*, where* $\check{H}^*$ *is Čech cohomology group,*$f^*$ *is induced by* $f$ *on Čech cohomology group.*

*Proof. theorem* 1,$H_{(n-p)}(X;G) \cong \check{H}^p(X;G)$. $\square$

**Corollary 2.** *let* $G = Z$*,then Čech pth cohomology group and Čech pth homology group is isomorphic, i.e.,*$\check{H}^p(X,Z) \cong H_p(X,Z)$.

**Corollary 3.** *In the sense of compact triangulable manifold space of $n$-dimension, Topological Fiber Entropy is true to homology group, i.e., $ent(f_L) \geq \log \|E_{f_* H_*(X;Z)}\|$, where $H_*(X;Z)$ is homology group.*

*Proof.* *Poincaré* Theorem[2] $P_{474}$ is valid on the compact homology $n-$ dimension manifold of triangulable, topology manifold is a subclass of homology manifold[2] $P_{462}$. □

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